Invariant Subspaces

representation $\varphi : G \longrightarrow GL(V)$

a vector subspace $W \leq V$ is

G-invariant if $\varphi_g(W) = W \quad \forall g \in G$

$\Longrightarrow$ get a representation

$$\varphi|_W : G \longrightarrow GL(W).$$

$w \in W \qquad (\varphi|_W)_g(w) := \varphi_g(w)$

called a subrepresentation of φ

Example: $\rho : S_n \to GL_n(\mathbb{C})$ std rep

$v = e_1 + \cdots + e_n = (1, 1, \ldots, 1)$,

$\forall g \in S_n, \quad \rho_g(v) = v$

$W := \mathbb{C}v = \{\lambda v\}$, invariant subspace

$\Longrightarrow \quad \rho|_W : S_n \to GL(W)$ 1-dim'l equivalent to trivial rep.

Example: $\rho : S_3 \to GL_3(\mathbb{C})$ std rep

$W = \mathbb{C}v$, $v = e_1 + e_2 + e_3$, invt subspace

$U = \mathbb{C}x + \mathbb{C}y$, $\quad x = e_1 - e_2$, $y = e_2 - e_3$ invt subspace

suffices to check $\rho_g(x), \rho_g(y) \in U$

$\Rightarrow \rho_g(U) \subseteq U$

for $g \in \{(12), (123)\}$

$x = e_1 - e_2$, $y = e_2 - e_3$

$\rho_{(12)}: \begin{array}{l} x \longrightarrow -x \\ y \longrightarrow x+y \end{array}$ $\qquad \rho_{(123)}: \begin{array}{l} x \longrightarrow y \\ y \longrightarrow -x-y \end{array}$

<u>In fact</u>: $W = \mathbb{C}v$, $U = \mathbb{C}x + \mathbb{C}y$

and $\{0\}$, $\mathbb{C}^3$ are the only

invariant subspaces of $\rho : S_3 \to GL_3(\mathbb{C})$